

Phys 235 Lecture Notes - Zonal Flows

Lecture by: P.H. Diamond, Notes by: Kurt Thompson

May 4, 2016

Lecture Dates: 4/12/16 - 4/14/16

Motivation to study Zonal Flows

We want to consider how small magnetic perturbations affect the transport of plasma in a uniform magnetic field. In the absence of the perturbations the guiding centers of the ions and electrons will travel along the field lines set up by the background unperturbed magnetic field, B_0 . In cases of spherical symmetry (such as toroidal or cylindrical symmetry), magnetic surfaces will be created where the guiding centers of the particles in the plasma are constrained to travel along these surfaces. In addition, poloidally symmetric shear flows called, zonal flows, will develop [1]-[2]. However if a magnetic perturbation is present that is sufficiently strong it can perturb the surfaces enough to where they overlap resulting in the destruction of the magnetic surfaces and the stochastic wandering of the particles. The magnetic perturbations can arise from two main sources: (1) self-consistent perturbations produced by a current density within the plasma and (2) perturbations introduced externally, i.e. via an external coil. The destruction of magnetic surfaces due to field irregularities has been well studied [3]-[4] as has the resulting particle transport due to the stochastic magnetic fields that arises from the destroyed magnetic surfaces [5].

With the objective of investigating particle transport due to magnetic perturbations we will consider how magnetic perturbations affect the drift kinetics of an electron distribution in a plasma. Assume that there are magnetic perturbations present that are sufficient enough to destroy the magnetic surfaces which results in particle diffusion. In addition, the magnetic perturbations will also cause the zonal flows to be perturbed. To see how these magnetic perturbations might affect zonal flows we start with the drift kinetic equation for electrons in a neutral plasma where it is assumed that there is a uniform magnetic field along the z (parallel/toroidal) direction and that there is a slight radial perturbation (\tilde{B}_r) to this magnetic field

$$\frac{\partial f}{\partial t} + (v_{\parallel} \cdot \nabla_{\parallel})f + v_{\parallel} \left(\frac{\tilde{B}_r}{B_0} \cdot \nabla_{\perp} \right) f + \dots = 0$$

where $f(\underline{x}, \underline{v})$ is the electron distribution function in position and velocity space and v_{\parallel} is the component of the electron velocity parallel to the unperturbed magnetic field. The

number density is related to the distribution function by integrating over all of velocity space

$$n(\underline{x}) = \int d\underline{v} f(\underline{x}, \underline{v})$$

Taking the ensemble average of the drift kinetic equation gives us

$$\frac{\partial \langle \tilde{n}_e \rangle}{\partial t} + \nabla_{\perp} \cdot \langle \tilde{v}_{\parallel} \frac{\tilde{B}_r}{B_0} n_0 \rangle + \dots = 0$$

where n_0 is the undisturbed electron number density, \tilde{n}_e is the perturbed electron number density and \tilde{v}_{\parallel} is the perturbed electron velocity in the z direction. The parallel electron velocity perturbation is related to the parallel electron current density perturbation via

$$\tilde{J}_{\parallel,e} = -|e|n_0\tilde{v}_{\parallel}$$

Substituting this into the averaged drift kinetic equation gives

$$\frac{\partial \langle \tilde{n}_e \rangle}{\partial t} + \nabla_{\perp} \cdot \left\langle \frac{\tilde{B}_r}{B_0} \frac{\tilde{J}_{\parallel,e}}{(-|e|)} \right\rangle + \dots = 0$$

Now we want to find another representation for $J_{\parallel,e}$. The Maxwell equation $\nabla \cdot B = 0$ implies that $B = \nabla \times A$ where A is the vector potential. Assuming that the change in the electric field is negligible then Ampere's law gives

$$\nabla \times B = \frac{4\pi J}{c}$$

$$\nabla \times (\nabla \times A) = \frac{4\pi J}{c}$$

$$\nabla(\nabla \cdot A) - \nabla^2 A = \frac{4\pi J}{c}$$

Using the Coulomb gauge where $\nabla \cdot A = 0$ we have

$$\nabla^2 A = -\frac{4\pi J}{c}$$

$$\Rightarrow \nabla_{\perp}^2 \tilde{A}_{\parallel} = -\frac{4\pi \tilde{J}_{\parallel}}{c}$$

$$\nabla_{\perp}^2 \tilde{A}_{\parallel} = -\frac{4\pi}{c}(\tilde{J}_{\parallel,e} + \tilde{J}_{\parallel,i})$$

$$\tilde{J}_{\parallel,e} = -\frac{c}{4\pi} \nabla_{\perp}^2 \tilde{A}_{\parallel} - \tilde{J}_{\parallel,i}$$

$$\tilde{J}_{\parallel,e} = -\frac{c}{4\pi} \nabla_{\perp}^2 \tilde{A}_{\parallel} - n_0 |e| \tilde{v}_{\parallel,i}$$

Substituting this result back into the averaged drift kinetic equation and ignoring the extra terms in the drift kinetic equation gives

$$\frac{\partial \langle \tilde{n}_e \rangle}{\partial t} + \nabla_{\perp} \cdot \left\langle \frac{\tilde{B}_r}{B_0} \left(\frac{c}{4\pi|e|} \nabla_{\perp}^2 \tilde{A}_{\parallel} + \frac{n_0|e|}{|e|} \tilde{v}_{\parallel,i} \right) \right\rangle = 0$$

$$\frac{\partial \langle \tilde{n}_e \rangle}{\partial t} = -\nabla_{\perp} \cdot \left\langle \frac{\tilde{B}_r}{B_0} \left(\frac{c}{4\pi|e|} \nabla_{\perp}^2 \tilde{A}_{\parallel} + n_0 \tilde{v}_{\parallel,i} \right) \right\rangle$$

By definition of the magnetic field in terms of the vector potential we have $\tilde{B}_r = \nabla_{\theta} \tilde{A}_{\parallel}$ and $-\tilde{B}_{\theta} = \nabla_r \tilde{A}_{\parallel}$. Substituting for \tilde{B}_r in the drift kinetic equation results in

$$\frac{\partial \langle \tilde{n}_e \rangle}{\partial t} = -\nabla_r \cdot \left\langle \frac{\nabla_{\theta} \tilde{A}_{\parallel}}{B_0} \left(\frac{c}{4\pi|e|} \nabla_{\perp}^2 \tilde{A}_{\parallel} + n_0 \tilde{v}_{\parallel,i} \right) \right\rangle$$

$$\frac{\partial \langle \tilde{n}_e \rangle}{\partial t} = -\frac{c}{4\pi|e|B_0} \nabla_r \cdot \langle (\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_{\perp}^2 \tilde{A}) \rangle - \frac{n_0}{B_0} \nabla_r \cdot \langle \tilde{B}_r \tilde{v}_{\parallel,i} \rangle$$

Looking at the first term on the right hand side we can use the Taylor identity calculation:

$$\langle (\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_{\perp}^2 \tilde{A}) \rangle = \langle (\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_r^2 \tilde{A} + \nabla_{\theta}^2 \tilde{A}) \rangle$$

The second term on the right hand side is zero because we are averaging over an odd number of ∇_{θ} terms, that is, these terms have odd k_{θ} moments in Fourier space

$$\Rightarrow \langle (\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_{\perp}^2 \tilde{A}) \rangle = \langle (\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_r^2 \tilde{A}) \rangle$$

$$\langle (\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_{\perp}^2 \tilde{A}) \rangle = \langle \nabla_r \cdot ((\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_r \tilde{A}_{\parallel})) \rangle - \langle (\nabla_r \cdot \nabla_{\theta} \tilde{A}_{\parallel})(\nabla_r \tilde{A}_{\parallel}) \rangle$$

$$\langle (\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_{\perp}^2 \tilde{A}) \rangle = \langle \nabla_r \cdot ((\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_r \tilde{A}_{\parallel})) \rangle - \langle (\nabla_{\theta} \cdot (\nabla_r \tilde{A}_{\parallel}))(\nabla_r \tilde{A}_{\parallel}) \rangle$$

Once again the second term on the right hand side is zero because we are averaging over an odd number of ∇_{θ} terms.

$$\langle (\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_{\perp}^2 \tilde{A}) \rangle = \langle \nabla_r \cdot ((\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_r \tilde{A}_{\parallel})) \rangle$$

$$\langle (\nabla_{\theta} \tilde{A}_{\parallel})(\nabla_{\perp}^2 \tilde{A}) \rangle = -\langle \nabla_r \cdot (\tilde{B}_r \tilde{B}_{\theta}) \rangle$$

Substituting this result back into the drift kinetic equation gives us

$$\frac{\partial \langle \tilde{n}_e \rangle}{\partial t} = \frac{c}{4\pi|e|B_0} \nabla_r^2 \langle (\tilde{B}_r \tilde{B}_{\theta}) \rangle - \frac{n_0}{B_0} \nabla_r \cdot \langle \tilde{B}_r \tilde{v}_{\parallel,i} \rangle$$

Looking at the terms in the brackets for this result we note that the first term on the right hand side is related to the magnetic stress since it has a $\tilde{B}_r \tilde{B}_{\theta}$ component. Whereas the second term depends on $\tilde{B}_r \tilde{v}_{\parallel,i}$ and therefore is like a magnetic "flutter" of parallel ion flow. There are three important key points to note regarding the final form of the drift kinetic equation: (1) no explicit dependence of the small electron inertia, (2) the presence of fluctuating parallel ion flows along with magnetic tilt (\tilde{B}_r) leads to a decrease in the

electron density (3) stochastic field perturbations affect the electron density by creating magnetic stresses. The third point suggests that stochastic fields have an effect on zonal flows since these flows are fundamentally a charge transport effect thus providing the motivation to study the effects of stochastic magnetic fields on zonal flows.

Zonal Flows

The simplest zonal flow model is the drift wave model (DW). We start by assuming that the divergence of the current density is zero and that in Fourier space $k_{\parallel} = 0$ and $k_{\theta} = 0$. In addition we will assume that there is a background unperturbed magnetic field, B_0 , and that there is no perturbing magnetic field.

$$\nabla \cdot J = 0$$

The current density can be broken into its parallel and perpendicular components

$$\nabla \cdot J = \nabla_{\perp} \cdot J_{\perp} + \nabla_{\parallel} \cdot J_{\parallel}$$

The second term on the right hand side is zero due to the condition $k_{\parallel} = 0$ and we can further separate J_{\perp} into its polarization current and global current

$$\nabla \cdot J = \nabla_{\perp} \cdot (J_{\perp,pol} + J_{\perp,cur})$$

where $J_{\perp,pol}$ is the current density due to the polarization and $J_{\perp,cur}$ is the current density due to the external field (global current). The second term on the right hand side is zero of the global current is zero since there is no perpendicular divergence of the unperturbed current.

$$\begin{aligned} \nabla \cdot J &= \nabla_{\perp} \cdot J_{\perp,pol} \\ \Rightarrow \nabla_{\perp} \cdot J_{\perp,pol} &= 0 \end{aligned}$$

This result is equivalent to the 2D vorticity equation of the form

$$\frac{\partial}{\partial t} \nabla_{\perp}^2 \phi + (\underline{v} \cdot \nabla) \nabla^2 \phi = 0$$

The first term can be thought of as the linear polarization drift and the second term is the nonlinear polarization drift. Now we average 2D vorticity equation over the zonal symmetry (i.e. poloidal symmetry)

$$\frac{\partial}{\partial t} \langle \nabla_r^2 \phi \rangle + \nabla_r \langle \tilde{v}_r \nabla_r^2 \phi \rangle = 0 \quad (1)$$

The first term in the brackets can be regarded as the polarization charge, Q_{pol} , via Poisson's equation and the second term in the brackets is the flux of the polarization charge.

$$\frac{\partial}{\partial t} \langle Q_{pol} \rangle + \nabla_r \langle \tilde{v}_r Q_{pol} \rangle = 0$$

Assuming a steady state solution where $\frac{\partial}{\partial t}\langle Q_{pol} \rangle = 0$ we must have that $\langle \tilde{v}_r Q_{pol} \rangle = 0$. Now we consider the net charge flux due to the ion perturbation, electron perturbation and polarization current:

$$\begin{aligned}\Gamma_{charge} &= |e|\langle \tilde{v}_r \tilde{n}_i \rangle + \langle \tilde{v}_r Q_{pol} \rangle - |e|\langle \tilde{v}_r \tilde{n}_e \rangle \\ \Rightarrow \Gamma_{charge} &= |e|\langle \tilde{v}_r \tilde{n}_i \rangle - |e|\langle \tilde{v}_r \tilde{n}_e \rangle\end{aligned}$$

For a neutral plasma with $\tilde{n}_i = \tilde{n}_e$ the two terms on the right hand side will cancel and we have

$$\Gamma_{charge} = 0$$

Thus we see that in the absence of magnetic perturbations the net charge flux will be zero. This means that without magnetic perturbations the zonal flows will be maintained and unperturbed.

Now we consider the same DW model except we introduce a radial magnetic perturbation and determine its effect on particle transport just like we did for the electron drift kinetic equation. In this case we still have that $\nabla \cdot J = 0$ but now $\nabla_{\perp} \cdot J_{\perp, pol} \neq 0$. Starting with the divergence of the current density

$$\begin{aligned}\nabla \cdot J &= \nabla_{\perp} \cdot J_{\perp} + \nabla_{\parallel} \cdot J_{\parallel} = 0 \\ \nabla \cdot J &= \nabla_{\perp} \cdot J_{\perp} + \nabla_{\parallel}^{(0)} \cdot J_{\parallel} + \frac{\tilde{B}_r}{B_0} \cdot \nabla_{\perp} J_{\parallel}\end{aligned}$$

As before the second term on the right hand side is zero because $k_{\parallel} = 0$ but the third term is not zero which leads to the relation

$$\nabla_{\perp} \cdot J_{\perp, pol} + \frac{\tilde{B}_r}{B_0} \cdot \nabla_{\perp} J_{parallel} = 0$$

This gives us a modified 2D vorticity equation:

$$\frac{\partial}{\partial t} \nabla_{\perp}^2 \phi + (\underline{v} \cdot \nabla) \nabla^2 \phi = \tilde{B}_r \cdot \nabla \tilde{J}_{\parallel}$$

where the term on the right hand side represents the current flow along the tilted field lines. Averaging over the zonal symmetry gives us

$$\begin{aligned}\frac{\partial}{\partial t} \langle \nabla_r^2 \phi \rangle + \nabla_r \langle \tilde{v}_r \nabla_r^2 \phi \rangle &= \nabla_r \langle \tilde{B}_r \tilde{J}_{\parallel} \rangle \\ \frac{\partial}{\partial t} \langle \nabla_r^2 \phi \rangle &= -\nabla_r \left(\langle \tilde{v}_r \nabla_r^2 \phi \rangle - \langle \tilde{B}_r \tilde{J}_{\parallel} \rangle \right)\end{aligned}\tag{2}$$

Comparing equation (2) with equation (1) we see that the effect of a radial magnetic perturbation creates a large effect on the polarization charges causing the current to flow along the tilted field lines. In doing so this will cause the zonal flows to be perturbed. Introducing the polarization charge as before we then have

$$\frac{\partial}{\partial t} \langle Q_{pol} \phi \rangle = -\nabla_r \left(\langle \tilde{v}_r Q_{pol} \phi \rangle - \langle \tilde{B}_r \tilde{J}_{\parallel} \rangle \right)$$

Assuming a steady state solution where $\frac{\partial}{\partial t}\langle Q_{pol} \rangle = 0$ we must have that $\langle \tilde{v}_r Q_{pol} \rangle = \langle \tilde{B}_r \tilde{J}_{\parallel} \rangle$. Assuming the neutral plasma condition ($\tilde{n}_i = \tilde{n}_e$) the net charge flux due to the ion perturbation, electron perturbation and polarization current is then given by

$$\Gamma_{charge} = |e|\langle \tilde{v}_r \tilde{n}_i \rangle + \langle \tilde{v}_r Q_{pol} \rangle - |e|\langle \tilde{v}_r \tilde{n}_e \rangle$$

$$\Gamma_{charge} = \langle \tilde{B}_r \tilde{J}_{\parallel} \rangle$$

From this we once again see that the presence of fluctuating parallel flows along with magnetic tilt (\tilde{B}_r) leads to a nonzero net flux of charge. This is equivalent to key point (2) that we noted regarding the analysis of the electron drift kinetic equation in the first section of this document.

Ku > 1 Regime

Up till now we have been discussing diffusion and particle transport in the regime where the Kubo number (Ku) is less than one, that is, $Ku < 1$. Now we want to start looking at the $Ku > 1$ regime. As an introductory example of $Ku > 1$ we will consider diffusion in a 2D Plasma and we will follow the treatment of J.B. Taylor and B. McNamara [6]. There are several reasons why we choose to analyze this problem as our canonical example of the $Ku > 1$ regime.

Firstly, this type of problem is fairly typical in experiments where a large uniform magnetic field inhibits the diffusion of plasma perpendicular to the field. A calculation of the perpendicular diffusion coefficient for a plasma in thermal equilibrium in this scenario will yield a coefficient that is proportional to $\frac{1}{B^2}$ [6]. However it turns out that, experimentally, the perpendicular diffusion coefficient is much larger and has a $\frac{1}{B}$ dependence (i.e. Bohm diffusion). Our treatment here of the 2D Plasma will derive perpendicular diffusion coefficients that are proportional to the Bohm diffusion coefficient which is more consistent with experimentation.

Secondly, we will find that the perpendicular diffusion coefficient will have a dependence on the system size. In other words, the coefficient will be an extensive quantity rather than an intensive quantity. This immediately leads us to reconsider if a diffusion treatment of the $Ku > 1$ regime is valid. As we will see in later lectures, the proper treatment of the 2D plasma requires percolation theory [7]. In addition, the 2D plasma is mathematically equivalent to the problem of transport in random media which is fundamentally a percolation problem. Therefore in the next lectures we will be interested in developing percolation theory in random media.

Before we discuss percolation theory in random medium we must first discuss the diffusion treatment of the 2D Plasma which necessarily leads us to percolation theory. To begin the discussion of the 2D Plasma we assume that a background unperturbed magnetic field, B_0 , has been set up in the z -direction and we are interested in the perpendicular diffusion, D_{\perp} , which we define as the time integral of the velocity correlation

$$D_{\perp} = \int_0^{\infty} d\tau \langle \tilde{v}(0) \tilde{v}(\tau) \rangle$$

$$D_{\perp} = \int_0^{\infty} d\tau \sum_k |\tilde{v}_k|^2 R(\tau)$$

where $R(\tau)$ is a memory function given by

$$\begin{aligned} R(\tau) &= e^{[-i\mathbf{k}\cdot\mathbf{r}_0 + i\mathbf{k}\cdot\mathbf{r}(-\tau)]} \\ \Rightarrow D_{\perp} &= \int_0^{\infty} d\tau \sum_k |\tilde{v}_k|^2 e^{[-i\mathbf{k}\cdot\mathbf{r}_0 + i\mathbf{k}\cdot\mathbf{r}(-\tau)]} \end{aligned}$$

We assume that the particles are only slightly perturbed from their orbits

$$\mathbf{r}(-\tau) = \mathbf{r}_0 + \delta\mathbf{r}(-\tau)$$

where $\delta\mathbf{r}$ is small compared to \mathbf{r}_0 . In addition we will assume that $\delta\mathbf{r}$ is stochastic so we need to use ensemble averaging

$$D_{\perp} = \int_0^{\infty} d\tau \sum_k |\tilde{v}_k|^2 \langle e^{i\mathbf{k}\cdot\delta\mathbf{r}(-\tau)} \rangle$$

We can expand the exponential

$$\langle e^{i\mathbf{k}\cdot\delta\mathbf{r}(-\tau)} \rangle = \left\langle \left(1 + i\mathbf{k}\cdot\delta\mathbf{r}(-\tau) - \frac{(\mathbf{k}\cdot\delta\mathbf{r}(-\tau))^2}{2!} + \dots \right) \right\rangle$$

The terms that are odd in k drop out and since $(\delta\mathbf{r})^2 \sim 2D_{\perp}\tau$ we have

$$\begin{aligned} \langle e^{-i\mathbf{k}\cdot\delta\mathbf{r}(-\tau)} \rangle &= \langle (1 - k_{\perp}^2 D_{\perp} \tau + \dots) \rangle \\ \langle e^{-i\mathbf{k}\cdot\delta\mathbf{r}(-\tau)} \rangle &= e^{-k_{\perp}^2 D_{\perp} \tau} \end{aligned}$$

Thus the perpendicular diffusion can be written as

$$\begin{aligned} D_{\perp} &= \int_0^{\infty} d\tau \sum_k |\tilde{v}_k|^2 e^{-k_{\perp}^2 D_{\perp} \tau} \\ D_{\perp} &= \sum_k |\tilde{v}_k|^2 \left[\frac{-1}{k_{\perp}^2 D_{\perp}} e^{-k_{\perp}^2 D_{\perp} \tau} \right]_0^{\infty} \\ D_{\perp}^2 &= \sum_k \frac{|\tilde{v}_k|^2}{k_{\perp}^2} \end{aligned}$$

If we assume a continuous symmetric k spectrum then we can exchange the summation over k for an integral over k -space where the limits are determined by the length scales of the problem (i.e. system size, Debye length, mean free path, etc.)

$$D_{\perp}^2 = \int_{k_{min}}^{k_{max}} dk_{\perp} k_{\perp} \frac{|\tilde{v}_k|^2}{k_{\perp}^2}$$

$$D_{\perp}^2 = \int_{k_{min}}^{k_{max}} dk_{\perp} \frac{|\tilde{v}_k|^2}{k_{\perp}}$$

The velocity perturbations are related to the electric field perturbations via $|\tilde{v}_k| = \frac{c}{B_0} |\tilde{E}_k|$ thus we have

$$D_{\perp}^2 = \frac{c^2}{B_0^2} \left[\int_{k_{min}}^{k_{max}} dk_{\perp} \frac{|\tilde{E}_k|^2}{k_{\perp}} \right]$$

This gives the general form of the perpendicular diffusion coefficient in a 2D Plasma. In order to proceed further, we must specify a type of electric field fluctuation spectrum. We will do this for two cases: (1) thermal equilibrium and (2) a random charge distribution. For thermal equilibrium the electric field fluctuation spectrum is given by [6]:

$$|E_k|^2 = \frac{4\pi k_B T}{l} \frac{1}{1 + k_{\perp}^2 \lambda_D^2}$$

where k_B is the Boltzmann constant, T is the temperature, λ_D^2 is the Debye length, l is the charge length in the z -direction. Substituting this into the equation for the perpendicular diffusion gives

$$\begin{aligned} D_{\perp}^2 &= \frac{c^2}{B_0^2} \left[\int_{k_{min}}^{k_{max}} dk_{\perp} \left(\frac{4\pi e}{el} \right) \frac{k_B T}{k_{\perp} (1 + k_{\perp}^2 \lambda_D^2)} \right] \\ D_{\perp}^2 &= \frac{c^2}{B_0^2} \left[\left(\frac{4\pi e^2 (k_B T)^2}{e^2 l k_B T} \right) \int_{k_{min}}^{k_{max}} dk_{\perp} \frac{1}{k_{\perp} (1 + k_{\perp}^2 \lambda_D^2)} \right] \\ D_{\perp} &= \frac{ck_B T}{eB_0} \left[\left(\frac{4\pi n e^2}{nl k_B T} \right) \int_{k_{min}}^{k_{max}} dk_{\perp} \frac{1}{k_{\perp} (1 + k_{\perp}^2 \lambda_D^2)} \right]^{\frac{1}{2}} \\ D_{\perp} &= \frac{ck_B T}{eB_0} \left[\left(\frac{1}{nl \lambda_D^2} \right) \int_{k_{min}}^{k_{max}} dk_{\perp} \frac{1}{k_{\perp} (1 + k_{\perp}^2 \lambda_D^2)} \right]^{\frac{1}{2}} \end{aligned}$$

Now use the trigonometric substitution $k_{\perp} = \frac{\tan \theta}{\lambda_D}$

$$\begin{aligned} D_{\perp} &= \frac{ck_B T}{eB_0} \left[\left(\frac{1}{nl \lambda_D^2} \right) \int \frac{\sec^2 \theta d\theta}{\lambda_D} \frac{\lambda_D}{\tan \theta (1 + \tan^2 \theta)} \right]^{\frac{1}{2}} \\ D_{\perp} &= \frac{ck_B T}{eB_0} \left[\left(\frac{1}{nl \lambda_D^2} \right) \int \frac{\cos \theta}{\sin \theta} \right]^{\frac{1}{2}} \\ D_{\perp} &= \frac{ck_B T}{eB_0} \left[\left(\frac{1}{nl \lambda_D^2} \right) [\ln |\sin \theta|] \right]^{\frac{1}{2}} \\ D_{\perp} &= \frac{ck_B T}{eB_0} \left[\left(\frac{1}{nl \lambda_D^2} \right) \left[\ln \left| \frac{k_{\perp}^2 \lambda_D^2}{[k_{\perp}^2 \lambda_D^2 + 1]^{\frac{1}{2}}} \right| \right]_{k_{min}}^{k_{max}} \right]^{\frac{1}{2}} \end{aligned}$$

For this system, k_{min} and k_{max} are given by

$$k_{min} \sim \frac{1}{L} \quad k_{max} \sim \frac{1}{\lambda_D}$$

where L is the system size. The perpendicular diffusion is then given by

$$D_{\perp} \sim \frac{ck_B T}{eB_0} \left[\left(\frac{1}{nl\lambda_D^2} \right) \left(\ln \left| \frac{1}{\sqrt{2}} \right| - \ln \left| \frac{\lambda_D/L}{[\lambda_D^2/L^2 + 1]^{\frac{1}{2}}} \right| \right) \right]^{\frac{1}{2}}$$

$$D_{\perp} \sim \frac{ck_B T}{eB_0} \left[\left(\frac{1}{2nl\lambda_D^2} \right) \left[\ln \left| \frac{1 + L^2/\lambda_D^2}{2} \right| \right] \right]^{\frac{1}{2}}$$

Assuming that $\frac{L^2}{\lambda_D^2} \gg 1$ we have

$$D_{\perp} \sim \frac{ck_B T}{eB_0} \left[\left(\frac{1}{nl\lambda_D^2} \right) \ln \left| \frac{L}{2\lambda_D} \right| \right]^{\frac{1}{2}} \quad (3)$$

From this result we see two important aspects. (1) If we identify the Bohm diffusion coefficient as $D_B = \frac{ck_B T}{eB_0}$ then we see that the perpendicular diffusion scales with the Bohm diffusion. (2) We also see that the perpendicular diffusion scales weakly with respect to the system size L and therefore the perpendicular diffusion is not an intensive parameter.

Now we will find the perpendicular diffusion for our second case which is a random array of charges. It will then be worthwhile to compare the perpendicular diffusion coefficients for the thermal equilibrium case and the case of a random array of charges. For a random array of point charges we can write the Poisson equation:

$$\begin{aligned} \nabla \cdot \underline{E} &= 4\pi\rho \\ \nabla \cdot \underline{E} &= \frac{4\pi}{l} \sum_i q_i \delta(x - x_i) \\ i\mathbf{k} \cdot \underline{E}_k &= \frac{4\pi}{l} \sum_i q_i e^{-i\mathbf{k} \cdot \mathbf{x}_i} \\ \Rightarrow |E_k|^2 &= \frac{1}{k_{\perp}^2} \left(\frac{4\pi}{l} \right)^2 \left\langle \sum_{i,j} q_i q_j e^{i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)} \right\rangle \\ &\Rightarrow |E_k|^2 = \frac{16\pi^2}{k_{\perp}^2 l^2} nq^2 \end{aligned}$$

Substituting this into the equation for the perpendicular diffusion gives

$$D_{\perp}^2 = \frac{c^2}{B_0^2} \left[\int_{k_{min}}^{k_{max}} dk_{\perp} \frac{16\pi^2}{k_{\perp}^3 l^2} nq^2 \right]$$

$$D_{\perp}^2 = \left(\frac{16\pi^2 c^2 n q^2}{l^2 B_0^2} \left[-\frac{1}{2k_{\perp}^2} \right]_{k_{min}}^{k_{max}} \right)$$

Assuming that $k_{min} \ll k_{max}$ we can approximate the perpendicular diffusion as

$$D_{\perp}^2 \sim \left(\frac{8\pi^2 c^2 n q^2}{l^2 B_0^2} \left[\frac{1}{k_{min}^2} \right] \right)$$

Assuming that k_{min} is related to the systems size as before, $k_{min} \sim \frac{1}{L}$, then we have

$$D_{\perp} \sim \left(\frac{2\sqrt{2}\pi c q n^{\frac{1}{2}}}{l B_0} L \right) \quad (4)$$

Here we see that the perpendicular diffusion has a strong dependence on the system size. Comparing equation (3) with (4) we see that the dependence on the systems size is much weaker in the thermal equilibrium case than in the case of a random array of charge. We also note that in both cases the perpendicular diffusion has a dependence on the systems size which, as mentioned earlier, leads us to reconsider if a diffusion treatment of $Ku > 1$ is correct and ultimately to consider percolation theory as the proper treatment of the 2D plasma.

Comparison of $Ku < 1$ and $Ku > 1$ Regimes

In the previous section we found that the general form of the perpendicular diffusion coefficient for the $Ku > 1$ regime is given by

$$\begin{aligned} D_{\perp}^2 &= \sum_k \frac{|\tilde{v}_k|^2}{k_{\perp}^2} \\ D_{\perp}^2 &\sim \sum_k \frac{|\tilde{B}_{r,k}|^2}{B_0^2 k_{\perp}^2} \\ D_{\perp}^2 &\sim \sum_k |\tilde{A}_k|^2 \\ D_{\perp} &\sim \left[\sum_k |\tilde{A}_k|^2 \right]^{\frac{1}{2}} \\ D_{\perp} &\sim \frac{\tilde{B}_{r,k} \Delta}{B_0} \end{aligned}$$

where Δ is the radial correlation length of the scattering particles. Following the treatment of Rosentbluth, et. al [4] the perpendicular diffusion coefficient for the $Ku < 1$ regime is given by

$$D_{\perp} \sim \sum_k \frac{|\tilde{B}_{r,k}|^2}{B_0^2} \delta(k_{\parallel}) \sim \frac{|\tilde{B}_r|^2}{B_0^2} l_{ac}$$

where l_{ac} is the autocorrelation length. Comparison of the diffusion coefficients for the two regimes shows that the perpendicular diffusion coefficient for the $Ku < 1$ regime is more sensitive to radial field perturbations than in the $Ku > 1$ limit. In addition, we see that for the $Ku > 1$ the particles will receive many kicks as the particle travels a distance of one autocorrelation length which leads to a diffusive process. For $Ku > 1$, the presence of Δ in the diffusion coefficient indicates that the particles will travel distance longer than Δ in one kick meaning that the scattering of the particles is very strong which again leads us to consider a percolation process rather than a diffusion process.

References

- [1] Itoh, K., Itoh, S.-I., Diamond, P. H., Hahm, T.S., Fujisawa, A., Tynan, G.R., Yagi, M. and Nagashima, Y., *Physics of Plasmas* 13, 055502 (2006).
- [2] Diamond, P. H., Itoh, S.-I., Itoh, K. and Hahm, T.S., *Plasma Phys. Control. Fusion* 47, R35 (2005).
- [3] Rechester, A.B., and Rosenbluth, M. N., *Phys Rev. Letters*, 40, 38 (1978).
- [4] Rosenbluth, M.N, Sagdeev, R. Z., Taylor, J.B. and Zaslavski, G.M., *Nuclear Fusion*, 6, 297 (1966).
- [5] Finn, J., Guzdar, P., and Chernikov, A., *Phys. Fluids B*, 4, 1152 (1992).
- [6] Taylor, J.B., and McNamara B., *Physics of Fluids*, 14, 1492 (1971).
- [7] Gruzinov, A.V., Isichenko, M.B., and Kalda, Ya. L., *Sov. Phys. JETP* 70 (2), February 1990.